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Hörmander multipliers on two-dimensional dyadic Hardy spaces

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ABSTRACT

In this paper we are interested in conditions on the coefficients of a two-dimensional Walsh multiplier operator that imply the operator is bounded on certain of the Hardy type spaces H^p, $0 < p < \infty$. We consider the classical coefficient conditions, the Marcinkiewicz–Hörmander–Mihlin conditions. They are known to be sufficient for the trigonometric system in the one and two-dimensional cases for the spaces L^p, $1 < p < \infty$. This can be found in the original papers of Marcinkiewicz [J. Marcinkiewicz, Sur les multiplicateurs des series de Fourier, Studia Math. 8 (1939) 78–91], Hörmander [L. Hörmander, Estimates for translation invariant operators in L^p spaces, Acta Math. 104 (1960) 93–140], and Mihlin [S.G. Mihlin, On the multipliers of Fourier integrals, Dokl. Akad. Nauk SSSR 109 (1956) 701–703; S.G. Mihlin, Multidimensional Singular Integrals and Integral Equations, Pergamon Press, 1965]. In this paper we extend these results to the two-dimensional dyadic Hardy spaces.

1. Introduction

We begin with some notation. Let r_k denote the kth Rademacher function, i.e.

$$r_0(x) = \begin{Bmatrix} +1 & \text{if } 0 \leqslant x < 1/2, \\ -1 & \text{if } 1/2 \leqslant x < 1 \end{Bmatrix}$$

of period 1, and

$$r_k(x) = r_0\left(2^k x\right) \quad (0 \leqslant x < 1,\ k \in \mathbb{N}).$$

The Walsh functions can be represented as products of Rademacher functions. Namely, if $n = \sum_{k=0}^{\infty} n_k 2^k$ ($n_k = 0$ or 1, $n \in \mathbb{N}$) is the binary decomposition of n then the nth Walsh function w_n in the Paley numeration is defined as

$$w_n = \prod_{k=0}^{\infty} r_k^{n_k}.$$

The dyadic expansion of $x \in [0, 1)$ is

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)} \quad (x_k = 0 \text{ or } 1).$$

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For the so-called dyadic rationals there are two expressions of this form. In this case we take the one which terminates in 0's. The dyadic addition ($\dot{+}$) is defined as follows

$$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)} \quad \big(x, y \in [0,1)\big).$$

Then by definition

$$w_k(x \dot{+} y) = w_k(x) w_k(y) \quad (k \in \mathbb{N},\ 0 \leqslant x, y < 1).$$

Let the space of functionals defined on $W = \{w_n\}_{n=0}^{\infty}$ be denoted by $F(W)$. Then the Walsh–Fourier coefficients, partial sums, and series of $f \in F(W)$ are defined by

$$\widehat{f}(k) = \langle f, w_k \rangle, \qquad S_n f = \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \qquad Sf = \sum_{k=0}^{\infty} \widehat{f}(k) w_k.$$

By means of the Walsh–Dirichlet kernels, $D_n = \sum_{k=0}^{n-1} w_k$ $(n \in \mathbb{N})$, and dyadic convolution the Walsh–Fourier partial sums can be can expressed as follows

$$S_n f(x) = (D_n * f)(x) = \int_0^1 f(t) D_n(x \dot{+} t)\, dt \quad \big(f \in L^1,\ 0 \leqslant x < 1\big).$$

The Kronecker product $w_{n,m}$ $(n, m = 0, 1, \ldots)$ of two Walsh systems is said to be the two-dimensional Walsh system. Thus

$$w_{n,m}(x, y) = w_n(x) w_m(y) \quad \text{and} \quad D_{n,m}(x, y) = D_n(x) D_m(y) \quad (n, m \in \mathbb{N},\ 0 \leqslant x, y < 0).$$

For the two-dimensional Walsh–Fourier coefficients of an integrable function the same notation will be used as in the one-dimensional case. That is

$$\widehat{f}(n, m) := \int_0^1 \int_0^1 f(x, y) w_{n,m}(x, y)\, dx\, dy \quad (n, m = 0, 1, \ldots).$$

The partial sum of the two-dimensional Walsh–Fourier series is given by

$$S_{n,m} f(x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \widehat{f}(i, j) w_{i,j}(x, y) = \int_0^1 \int_0^1 f(t, u) D_n(x \dot{+} t) D_m(y \dot{+} u)\, dt\, du.$$

In the special case $n = 2^k$, see e.g. [13] or [15], we have

$$D_{2^k} = 2^k \chi_{[0, 2^{-k})}, \tag{1.1}$$

where χ_A is the characteristic function of $A \subset [0, 1)$. Consequently,

$$S_{2^k, 2^l} f(x, y) = 2^{k+l} \int_{[0, 2^{-k}) \times [0, 2^{-l})} f,$$

i.e. $S_{2^k, 2^l} f$ is the average value of f over the dyadic rectangle $[0, 2^{-k}) \times [0, 2^{-l})$. Intervals of the form $[k 2^{-n}, (k+1) 2^{-n})$ $(k, n \in \mathbb{N},\ k < 2^{-n})$ are called dyadic intervals.

The Dirichlet kernels can be decomposed in several ways. The one we will use in the proofs is the following (see e.g. [15])

$$D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} \quad (n \in \mathbb{N}). \tag{1.2}$$

The Hardy spaces H^p are central to the development of Walsh–Fourier analysis. One can use several models like dyadic martingales, quasi-measures, formal Walsh series, dyadic distributions etc. for representing the elements of H^p. We will stay with the simple model that f is a functional defined on Walsh functions, i.e. $f \in F(W)$. The maximal function and the square function are defined as

$$f^* = \sup_{n,m} |S_{2^n, 2^m} f|,$$

$$Qf = \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |S_{2^n, 2^m} f - S_{2^{n-1}, 2^m} f - S_{2^n, 2^{m-1}} + S_{2^{n-1}, 2^{m-1}} f|^2 \right)^{1/2},$$

where $S_{2^{-1},2^m} f = S_{2^n,2^{-1}} f = 0$ $(n, m = -1, 0, 1, \ldots)$. Then f is said to belong to H^p if $f^* \in L^p$ and the norm is defined as $\|f\|_{H^p} = \|f^*\|_p$ $(0 < p < \infty)$. Moreover it is known that $\|f^*\|_p \approx \|Qf\|_p$ $(0 < p < \infty)$.

We will use another type of two-dimensional dyadic Hardy spaces for $p = 1$. Assume that f is a measurable function. Set $f_y(x) = f(x, y)$ $(0 \leqslant x, y < 1)$. Thus $H_\sharp$ is defined by the so-called hybrid maximal function

$$f^\sharp(x, y) = \left|\sup_n S_{2^n} f_y(x)\right|, \qquad \|f\|_{H_\sharp} = \left\|f^\sharp\right\|_1.$$

The hybrid square function is $Q_\sharp f(x, y) = (\sum_{k=0}^\infty |S_{2^k} f_y(x) - S_{2^{k-1}} f_y(x)|^2)^{1/2}$.

We will show the boundedness of multiplier operators from Hardy–Lorentz spaces to Lorentz spaces. Lorentz spaces $L^{p,q}$ are defined via

$$\|f\|_{p,q} = \left(\int_0^\infty \widetilde{f}(t)^q t^{t/q} \frac{dt}{t}\right)^{1/q} \quad (0 < p, q < \infty),$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} \widetilde{f}(t) \quad (0 < p \leqslant \infty),$$

where f is a measurable function, and $\widetilde{f}$ is the non-increasing rearrangement of f. For properties of Lorentz spaces see e.g. Bennett and Sharpley [1]. Hardy–Lorentz spaces, $H^{p,q}$, $H_\sharp^{p,q}$ are defined in the same manner with the modification that f is replaced by the corresponding square function.

2. Main result

In this paper we will investigate multiplier operators T_ϕ where ϕ is a bounded sequence $(\phi(k, \ell)\colon k, \ell = 0, 1, 2, \ldots)$ of numbers and $\widehat{T_\phi f}(k, \ell) = \phi(k, \ell)\widehat{f}(k, l)$. As ϕ is bounded, T_ϕ is bounded on L^2. Marcinkiewicz in 1939 [9] published the seminal paper concerning multiplier operators in one and two dimensions for classical Fourier series. He considered the case of Lebesgue spaces L^p, $1 < p < \infty$, and showed the multiplier operator was bounded on these spaces if the multiplier satisfies the difference condition on the multiplier coefficients that has come to be called the Marcinkiewicz condition. For the Euclidean versions, see papers [8,10,11]. In one dimension this condition is

$$\sum_{k=2^n}^{2^{n+1}-1} \left|\phi(k) - \phi(k+1)\right| \leqslant B.$$

The related Marcinkiewicz–Hörmander–Mihlin (M–H–M) condition is

$$\sum_{k=2^n}^{2^{n+1}-1} \left|\phi(k) - \phi(k+1)\right|^q \leqslant B 2^{n(1-q)} \tag{2.1}$$

for $q > 1$ and some constant B independent of $n = 0, 1, 2, \ldots$. The authors have studied these conditions extensively in the context of Hardy spaces on the dyadic group [2], Vilenkin groups [5], dyadic field [4], and the classical case [3]. In two dimensions, set

$$\Delta^{(1)}\phi(u, v) = \phi(u, v) - \phi(u+1, v), \qquad \Delta^{(2)}\phi(u, v) = \phi(u, v) - \phi(u, v+1),$$

and

$$\Delta\phi(u, v) = \phi(u, v) - \phi(u+1, v) - \phi(u, v+1) + \phi(u+1, v+1).$$

Then the two-dimensional Marcinkiewicz–Hörmander–Mihlin condition is: for each $j, k \in \mathbb{N}$ and $1 \leqslant q < \infty$,

$$\sum_{u=2^{j-1}}^{2^j-1} \left|\Delta^{(1)}\phi\big(u, 2^k - 1\big)\right|^q \leqslant C\, 2^{j(1-q)}, \qquad \sum_{v=2^{k-1}}^{2^k-1} \left|\Delta^{(2)}\phi\big(2^j - 1, v\big)\right|^q \leqslant C\, 2^{k(1-q)},$$
$$\sum_{u=2^{j-1}}^{2^j-1} \sum_{v=2^{k-1}}^{2^k-1} \left|\Delta\phi(u, v)\right|^q \leqslant C\, 2^{(j+k)(1-q)}. \tag{2.2}$$

The main result of this work is the following:

Theorem 2.1. *If the bounded multiplier ϕ satisfies the two-dimensional Marcinkiewicz–Hörmander–Mihlin condition* (2.2) *for some q, $1 < q \leqslant 2$, and if $p > \frac{q}{2q-1}$, then T_ϕ is bounded from $H^{p,r}$ to $L^{p,r}$ $(0 < r \leqslant \infty)$. Moreover, T_ϕ is of weak type $(H_\sharp, L^1)$.*

This result is known for $p > 1$, when $H^{p,r} \sim L^{p,r}$, so we only need to deal with the case $\frac{q}{2q-1} < p \leqslant 1$. Note that as q tends to 2, the bound for p tends to $2/3$ as $p > \frac{q}{2q-1}$. This is consistent with the one-dimensional version of Theorem 2.1 shown in [2]. At the end of the paper we present a simplified proof in the case the two-dimensional multiplier is the product of two one-dimensional multipliers each of which satisfies the one-dimensional M–H–M condition.

Since $H^{p,p} = H^p$, and $L^{p,p} = L^p$ we have from Theorem 2.1 that, in particular, T_ϕ is bounded from H^p to L^p for $\frac{q}{2q-1} < p \leqslant 1$. By means of the Khinchin inequality one can prove, see e.g. [16] for the reasoning, that if a multiplier operator is bounded from H^p to L^p then it is also bounded on H^p. We note that in other terminology this is a consequence of the fact that multiplier operators commute with conjugation, the concept of which was introduced by Weisz (see [20]).

Corollary 2.2. *If the bounded multiplier ϕ satisfies the two-dimensional Marcinkiewicz–Hörmander–Mihlin condition* (2.2) *for some q, $1 < q \leqslant 2$, and if $p > \frac{q}{2q-1}$, then T_ϕ is bounded on H^p.*

The sharpness of our result follows easily from the example given in our paper [2] for the one-dimensional case. Namely, we showed that if $1 \leqslant r \leqslant 2$, and $p < r/(2r-1)$ then there exist a bounded multiplier $\lambda = (\lambda_k)$, and an f in the one-dimensional H^p space such that λ satisfies (2.1) but $T_\lambda f$ is not in L^p. Set $\phi(k,0) = \lambda_k$, and $\phi(k,\ell) = 0$ for $\ell \neq 0$. Then as is easy to see ϕ satisfies (2.2). Moreover let $h(x,y) = f(x)$, $0 \leqslant x, y < 1$. Then $h \in H^p$, and $T_\phi h \notin L^p$.

Corollary 2.3. *Theorem* 2.1 *is sharp in the following sense. If $1 < r \leqslant 2$, and $p < r/(2r-1)$ then the statements* (i) *and* (ii) *in Theorem* 2.1 *do not hold. In particular there exists a bounded multiplier ϕ that satisfies* (2.2) *but T_ϕ is not bounded from H^p to L^p.*

Remark 2.4. We do not know what is the situation in the endpoint case. We note that for the one-dimensional Fourier transform and the Marcinkiewicz condition Tao and Wright proved in [17] that the Marcinkiewicz multipliers are bounded from H^1 to $L^{1,\infty}$, i.e. to the weak L^1 space, but this does not hold for any $L^{1,r}$ with $r < \infty$.

In the proof of Theorem 2.1 we will actually show that T_ϕ is H^p-quasilocal operator. As it was shown by Weisz in [18] this implies Theorem 2.1. Generally speaking proofs of results on Hardy spaces are often based on the atomic structure of these spaces. In two dimensions, the structure of Hardy spaces is more complicated than in the one-dimensional case (see Weisz [18]). This is reflected also in the atomic structure. In two dimensions, the support of an atom need not be a dyadic rectangle but can be a general open set. This makes the atomic structure more difficult to use. However, it is sufficient to consider only special types of atoms—the rectangular atoms—to show the boundedness of certain operators. To this end, let $0 < p \leqslant 1$. A function $a \in L^2$ is called a rectangular H^p-atom if either a is identically equal to 1 or there exists a dyadic rectangle I such that

$$\operatorname{supp}(a) \subset I, \qquad \|a\|_2 \leqslant |I|^{1/2-1/p},$$
$$\int_0^1 a(x,t)\,dt = \int_0^1 a(u,y)\,du = 0 \quad \big(x, y \in [0,1)\big).$$

Although the elements of H^p cannot be decomposed into sums of rectangular atoms, in the case of H^p-quasilocal operators, it is sufficient to consider the action of such operators on individual rectangular atoms.

To define the p-quasilocality of an operator T, we will assume T is sublinear and bounded from L^2 to L^2 (see also Simon [16]). Then T is called H^p-quasilocal if there exists $\delta > 0$ such that for every rectangular H^p-atom a supported on the dyadic rectangle I and for all $r = 0, 1, 2, \ldots,$ the estimate below holds:

$$\int_{[0,1)^2 \setminus I^r} |Ta|^p \leqslant C_p 2^{-\delta r}. \tag{2.3}$$

Here I^r is the dyadic rectangle defined as follows: $I^r = I_1^r \times I_2^r$, where $I = I_1 \times I_2$ for dyadic intervals I_1, I_2, and I_j^r is the unique dyadic interval for which $I_j \subset I_j^r$ and $\frac{|I^r|}{|I|} = 2^r$ $(j = 1, 2)$.

We will need both a one and two-dimensional Sidon-type inequalities to handle the estimation of the linear combinations of Dirichlet kernels that arise in the estimation of these operators. These results are included in Lemma 3.1.

Before proceeding to the proofs of the main theorem and Sidon type inequalities, we will apply these results to specific multipliers that have been considered by other authors. Marcinkiewicz in his original paper, applied his theorem to a number of multipliers. Namely,

$$\phi_1(u,v) = \frac{u^2}{u^2+v^2}, \qquad \phi_2(u,v) = \frac{uv}{u^2+v^2}, \qquad \phi_3(u,v) = \frac{v^2}{u^2+v^2}.$$

The differences for these three multipliers all satisfy the same estimates. For $i = 1, 2, 3$, a direct computation shows that

$$\left|\Delta^{(1)}\phi_i\left(u, 2^j\right)\right| \leqslant \frac{C}{\sqrt{u^2 + 2^{2j}}}, \qquad \left|\Delta^{(2)}\phi_i\left(2^k, v\right)\right| \leqslant \frac{C}{\sqrt{2^{2k} + v^2}}, \qquad \left|\Delta\phi_i(u, v)\right| \leqslant \frac{5}{u^2 + v^2}.$$

This implies directly each of the multipliers satisfies the conditions of Theorem 2.1 for all q, $1 < q \leqslant 2$, and thus the corresponding multiplier operators are bounded on H^p for $2/3 < p \leqslant 1$. The two-dimensional Sunouchi multiplier and its inverse have been the subject of papers by both Weisz [19] and Simon [16]. These multipliers are connected to the characterization of the Hardy spaces H^p by the Sunouchi square function. The two-dimensional Sunouchi multiplier ϕ is given by

$$\phi(u, v) = \frac{uv}{2^{n+m}}$$

for $2^{n-1} \leqslant u < 2^n$ and $2^{m-1} \leqslant v < 2^m$. Again, the Sunouchi multiplier and its inverse satisfy the theorem for all q, $1 < q < \infty$. So by Theorem 2.1, the multiplier operator is bounded on H^p for $2/3 < p \leqslant 1$. Simon [16] proved that the Sunouchi multiplier is bounded on H^p, for $p > 0$ and the inverse for $p > 1/2$, which are the same results as obtained by Daly and Phillips [6] in the one-dimensional case. Thus the bounds on p in our main result are in some sense not optimal. The general M–H–M condition does not give as precise results as those obtained for this pair of multipliers by Simon, Daly and Phillips, but this is not unexpected. Concerning the properties of multiplier operators, in particular the Marcinkiewicz and the Sunouchi multipliers, with respect to the Ciesielski system we call the attention to two recent papers by Weisz [21,22].

3. Proofs of the main results

Lemma 3.1 *(Sidon-type inequalities). Let* $1 < q \leqslant 2$ *and* $n, m, N, M \in \mathbb{N}$*. Then*

$$\text{(i)}\quad \int_{2^{-N}}^{1}\left|\sum_{k=1}^{2^n} c_k D_k(x)\right| dx \leqslant C 2^{N(1-\frac{1}{q})}\left(\sum_{k=1}^{2^n}|c_k|^q\right)^{1/q},$$

$$\text{(ii)}\quad \int_{2^{-N}}^{1}\int_{2^{-M}}^{1}\left|\sum_{k=1}^{2^n}\sum_{j=1}^{2^m} c_{k,j} D_{k,j}(x, y)\right| dx\,dy \leqslant C 2^{(N+M)(1-\frac{1}{q})}\left(\sum_{k=1}^{2^n}\sum_{j=1}^{2^m}|c_{k,j}|^q\right)^{1/q},$$

$$\text{(iii)}\quad \int_{2^{-N}}^{1}\left|\sum_{k=1}^{2^n} c_k D_k(x)\right|^2 dx \leqslant C 2^{N(3-2/q)}\left(\sum_{k=1}^{2^n}|c_k|^q\right)^{2/q}.$$

Proof. We provide a proof only of (ii) and (iii). The proof of inequality (i) is found in [2]. Its trigonometric equivalent was proved by Móricz in [12], and the nontruncated dyadic version was given by Schipp in [14]. We first estimate the integrand of the left side in the statement of (ii). By using the decomposition in (1.2) we have that

$$\left|\sum_{k=1}^{2^m}\sum_{j=1}^{2^n} c_{k,j} D_{k,j}(x, y)\right| = \left|\sum_{k=1}^{2^m}\sum_{j=1}^{2^n} c_{k,j} D_k(x) D_j(y)\right| = \left|\sum_{k=1}^{2^m}\sum_{j=1}^{2^n} c_{k,j} w_k(x) w_j(y)\sum_{\mu=0}^{N-1} k_\mu D_{2^\mu}(x)\sum_{\upsilon=0}^{M-1} k_\upsilon D_{2^\upsilon}(y)\right|$$

holds for any $2^{-M} \leqslant x < 1$, and $2^{-N} \leqslant y < 1$. We note that the inner sums stop at $(N-1)$ and $(M-1)$ respectively as the integrals are over the intervals $[2^{-N}, 1)$ and $[2^{-M}, 1)$. Continuing,

$$\begin{aligned}
&\leqslant \sum_{\mu=0}^{N-1}\sum_{\upsilon=0}^{M-1} 2^\mu 2^\upsilon \chi_{[0,2^{-\mu})}(x)\chi_{[0,2^{-\upsilon})}(y)\left|\sum_{k=1}^{2^n}\sum_{j=1}^{2^m} c_{k,j} k_\mu k_\upsilon w_k(x) w_j(y)\right| \\
&= \sum_{\mu=0}^{N-1}\sum_{\upsilon=0}^{M-1} 2^\mu 2^\upsilon \chi_{[0,2^{-\mu})}(x)\chi_{[0,2^{-\upsilon})}(y) h_{\mu,\upsilon}(x, y)\sum_{k=1}^{2^n}\sum_{j=1}^{2^m} c_{k,j} k_\mu k_\upsilon w_k(x) w_j(y),
\end{aligned}$$

where $h_{\mu,\upsilon}(x, y) = \operatorname{sgn}(\sum_{k=1}^{2^n}\sum_{j=1}^{2^m} c_{k,j} k_\mu k_\upsilon w_k(x) w_j(y))$. Thus the integral on the left-hand side of the statement of the lemma is bounded by

$$\begin{aligned}
&\leqslant \sum_{\mu=0}^{N-1}\sum_{\upsilon=0}^{M-1} 2^\mu 2^\upsilon \sum_{k=1}^{2^n}\sum_{j=1}^{2^m} c_{k,j} k_\mu k_\upsilon \int_0^1\int_0^1 \chi_{[0,2^{-\mu})}(x)\chi_{[0,2^{-\upsilon})}(y) h_{\mu,\upsilon}(x, y) w_k(x) w_j(y)\,dx\,dy \\
&= \sum_{\mu=0}^{N-1}\sum_{\upsilon=0}^{M-1} 2^\mu 2^\upsilon \sum_{k=1}^{2^n}\sum_{j=1}^{2^m} c_{k,j} k_\mu k_\upsilon \left(\chi_{[0,2^{-\mu})\times[0,2^{-\upsilon})} h_{\mu,\upsilon}\right)^\wedge(k, j).
\end{aligned}$$

For the inner two sums, we first use Hölder's inequality followed by the Hausdorff–Young theorem to obtain:

$$\begin{aligned}
&\left|\sum_{k=1}^{2^n}\sum_{j=1}^{2^m} c_{k,j}k_\mu k_\upsilon\big(\chi_{[0,2^{-\mu})\times[0,2^{-\upsilon})}h_{\mu,\upsilon}\big)\hat{}(k,j)\right| \\
&\quad\leqslant \left(\sum_{k=1}^{2^n}\sum_{j=1}^{2^m}|c_{k,j}|^q\right)^{1/q}\left(\sum_{k=1}^{2^n}\sum_{j=1}^{2^m}\left|\big(\chi_{[0,2^{-\mu})\times[0,2^{-\upsilon})}h_{\mu,\upsilon}\big)\hat{}(k,j)\right|^{q'}\right)^{1/q'} \\
&\quad\leqslant \left(\sum_{k=1}^{2^n}\sum_{j=1}^{2^m}|c_{k,j}|^q\right)^{1/q}\|\chi_{[0,2^{-\mu})\times[0,2^{-\upsilon})}h_{\mu,\upsilon}\|_q \leqslant \left(\sum_{k=1}^{2^n}\sum_{j=1}^{2^m}|c_{k,j}|^q\right)^{1/q} 2^{-(\mu+\upsilon)/q}.
\end{aligned}$$

Substituting this estimate for the inner two sums above gives the estimate

$$\begin{aligned}
&\leqslant C\sum_{\mu=0}^{N-1}\sum_{\upsilon=0}^{M-1}2^\mu 2^\upsilon 2^{-(\mu+\upsilon)/q}\left(\sum_{k=1}^{2^n}\sum_{j=1}^{2^m}|c_{k,j}|^q\right)^{1/q} \\
&\leqslant C2^{(N+M)(1-\frac{1}{q})}\left(\sum_{k=1}^{2^n}\sum_{j=1}^{2^m}|c_{k,j}|^q\right)^{1/q},
\end{aligned}$$

which is the desired estimate.

For (iii), let us decompose the sum $\sum_{k=1}^{2^n} c_k D_k$ into blocks as follows:

$$\sum_{k=1}^{2^n} c_k D_k = \sum_{\ell=0}^{2^{n-N}-1}\ \sum_{\ell 2^N+1}^{(\ell+1)2^N} c_k D_k.$$

The blocks can be further decomposed as

$$\sum_{\ell 2^N+1}^{(\ell+1)2^N} c_k D_k = \sum_{k=1}^{2^N} c_{\ell 2^N+k} D_{\ell 2^N} + w_{\ell 2^N}\sum_{k=1}^{2^N} c_{\ell 2^N+k} D_k.$$

Clearly $D_{\ell 2^N} = \sum_{j=0}^{\ell-1} w_{j2^N} D_{2^N}$. Since D_{2^N} is 0 outside the interval $[0, 2^{-N}]$ we have that

$$\sum_{\ell 2^N+1}^{(\ell+1)2^N} c_k D_k(x) = w_{\ell 2^N}(x)\sum_{k=1}^{2^N} c_{\ell 2^N+k} D_k(x) \quad \big(2^{-N}\leqslant x\leqslant 1\big).$$

Let us take two blocks corresponding to $j\neq\ell$ ($0\leqslant j,\ell<2^{n-N}$). By the definition of Walsh functions we have that w_{j2^N} and $w_{\ell 2^N}$ are orthogonal on every dyadic interval of length 2^{-N}. On the other hand, both $\sum_{k=1}^{2^N} c_{\ell 2^N+k}D_k$ and $\sum_{k=1}^{2^N} c_{j2^N+k}D_k$ are constants on such intervals. Consequently the different blocks are orthogonal on $[2^{-N},1]$. Hence

$$\int_{2^{-N}}^{1}\left(\sum_{k=1}^{2^n} c_k D_k\right)^2 = \sum_{\ell=0}^{2^{n-N}-1}\int_{2^{-N}}^{1}\left(\sum_{k=1}^{2^N} c_{\ell 2^N+k} D_k\right)^2.$$

It is known (see [7] or [15]), that $|D_k(x)|\leqslant\frac{2}{x}$ ($k\in\mathbb{N}$, $0<x\leqslant 1$). Therefore,

$$\begin{aligned}
\int_{2^{-N}}^{1}\left(\sum_{k=1}^{2^N} c_{\ell 2^N+k} D_k\right)^2 &\leqslant 4\left(\sum_{k=1}^{2^N}|c_{\ell 2^N+k}|\right)^2\int_{2^{-N}}^{1}\frac{1}{x^2}\,dx \leqslant 4\left(2^{N(1-1/q)}\left(\sum_{k=1}^{2^N}|c_{\ell 2^N+k}|^q\right)^{1/q}\right)^2\cdot 2^N \\
&= 4\cdot 2^{N(3-2/q)}\left(\sum_{k=1}^{2^N}|c_{\ell 2^N+k}|^q\right)^{2/q} \quad \big(0\leqslant\ell<2^{n-N}\big).
\end{aligned}$$

Since $2/q\geqslant 1$ we have

$$\int_{2^{-N}}^{1}\left(\sum_{k=1}^{2^n} c_k D_k\right)^2 \leqslant 4\cdot 2^{N(3-2/q)}\sum_{\ell=0}^{2^{n-N}-1}\left(\sum_{k=1}^{2^N}|c_{\ell 2^N+k}|^q\right)^{2/q} \leqslant 4\cdot 2^{N(3-2/q)}\left(\sum_{k=0}^{2^n}|c_k|^q\right)^{2/q},$$

which was to be proved. □

We will need the following technical lemma in the subsequent work. This shows that the two-dimensional M–H–M condition (2.2) implies the one-dimensional M–H–M condition on strips.

Lemma 3.2. *The two-dimensional M–H–M condition* (2.2) *implies that*

$$\sum_{u=2^{j-1}}^{2^j-1} \left|\Delta^{(1)}\phi(u,\ell)\right|^q \leqslant C\,2^{j(1-q)} \quad \text{and} \quad \sum_{v=2^{k-1}}^{2^k-1} \left|\Delta^{(2)}\phi(\ell,v)\right|^q \leqslant C\,2^{k(1-q)}$$

hold for every $l \in \mathbb{N}$.

Proof. We will show that the second inequality holds. The proof of the first one follows in exactly the same manner.

Let $2^{j-1}-1 < \ell < 2^j$. We may suppose $j > 1$. Then

$$\Delta^{(2)}\phi(\ell,v) = \Delta^{(2)}\phi\big(2^{j-1}-1,v\big) - \Delta\phi\big(2^{j-1}-1,v\big) - \sum_{u=2^{j-1}}^{\ell-1} \Delta\phi(u,v).$$

Hence

$$\begin{aligned}\left(\sum_{v=2^{k-1}}^{2^k-1} \left|\Delta^{(2)}\phi(\ell,v)\right|^q\right)^{1/q} &\leqslant \left(\sum_{v=2^{k-1}}^{2^k-1} \left|\Delta^{(2)}\phi\big(2^{j-1}-1,v\big)\right|^q\right)^{1/q} \\ &\quad + \left(\sum_{v=2^{k-1}}^{2^k-1} \left|\Delta\phi\big(2^{j-1}-1,v\big)\right|^q\right)^{1/q} + \left(\sum_{v=2^{k-1}}^{2^k-1} \left|\sum_{u=2^{j-1}}^{\ell-1} \Delta\phi(u,v)\right|^q\right)^{1/q} \\ &= A_1 + A_2 + A_3.\end{aligned}$$

By (2.2), we have $A_1 \leqslant C\,2^{k(1/q-1)}$. If $j = 2$ then the same estimate follows for A_2 directly from (2.2). If $j > 2$ then again by (2.2) and by Hölder's inequality for the inner sum we obtain

$$\begin{aligned}A_2 &\leqslant \left(\sum_{v=2^{k-1}}^{2^k-1} \left(\sum_{u=2^{j-2}-1}^{2^{j-1}-1} \left|\Delta\phi(u,v)\right|\right)^q\right)^{1/q} \leqslant C\,2^{j(1-1/q)} \left(\sum_{v=2^{k-1}}^{2^k-1} \sum_{u=2^{j-2}-1}^{2^{j-1}-1} \left|\Delta\phi(u,v)\right|^q\right)^{1/q} \\ &\leqslant C\,2^{j(1-1/q)} 2^{(j+k)(1/q-1)} \leqslant C\,2^{k(1/q-1)}.\end{aligned}$$

In the same way

$$A_3 \leqslant \left(\sum_{v=2^{k-1}}^{2^k-1} \left(\sum_{u=2^{j-1}}^{2^j-1} \left|\Delta\phi(u,v)\right|\right)^q\right)^{1/q} \leqslant C\,2^{k(1/q-1)}.$$

The desired estimate holds for each of the three terms. $\square$

Proof of Theorem 2.1. We will show the operator T_ϕ is p-quasilocal. As T_ϕ is translation invariant, we need only consider dyadic rectangles of the form $I = [0,2^{-N}) \times [0,2^{-M})$. We will be integrating over the complements of dilates of I: for $r \in \mathbb{N}$,

$$\begin{aligned}\big(I^r\big)^c &= \big(\big[0,2^{-N+r}\big) \times \big[0,2^{-M+r}\big)\big)^c \\ &= \big[0,2^{-N+r}\big) \times \big[2^{-M+r},1\big) \cup \big[2^{-N+r},1\big) \times \big[2^{-M+r},1\big) \cup \big[2^{-N+r},1\big)\big[0,2^{-M+r}\big) \\ &= (1) \cup (2) \cup (3).\end{aligned}$$

Set (2) is the product of two complements of intervals about 0 in one dimension. The argument for boundedness in this case will be an iterated version of the one-dimensional proof. Sets (1) and (3) are symmetric in x and y, so we need only provide the proof for one of the sets. As each of these sets has a neighborhood of 0 as one of the factors, the proof in these cases will be more involved and require additional techniques as compared to the one-dimensional case. The arguments of Simon for the two-dimensional Sunouchi operator do not apply here. The Sunouchi multiplier splits into the product of two one-dimensional multipliers and Simon's argument uses this fact in an essential manner. At the end of the paper, we will provide a simplified proof for the special case in which this factorization occurs. This is Theorem 3.3.

Let $a \in L^2$, $\operatorname{supp} a \subset I = [0,2^{-N}) \times [0,2^{-M})$, and $\|a\|_2 \leqslant |I|^{1/2-1/p}$. Note that both rectangle H^p-atoms and square $H^p_\sharp$-atoms are included. Then

$$\int_{(I^r)^c} |T_\phi a|^p = \int_{[0,2^{-N+r})\times[2^{-M+r},1)} |T_\phi a|^p + \int_{[2^{-N+r},1)\times[2^{-M+r},1)} |T_\phi a|^p + \int_{[2^{-N+r},1)\times[0,2^{-M+r})} |T_\phi a|^p = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.$$

Recall that $a \in L^2$ and ϕ is bounded. Therefore expanding $T_\phi a$ into a Walsh series we have $S(T_\phi a) = T_\phi a$ both a.e. and in L^2. In particular

$$S_{2^K,2^L}(T_\phi a) = \sum_{i=N}^{K-1} \sum_{j=M}^{L-1} \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \widehat{a}(k,\ell)\phi(k,\ell) w_{k,\ell} \to T_\phi a \tag{3.1}$$

a.e. and in L^2 as $k, \ell \to \infty$. The sums begin at 2^N and 2^M as the atom a has integral zero with support $I = [0, 2^{-N}) \times [0, 2^{-M})$. We note that the convergence relation in (3.1) justifies part of our manipulations below.

We begin with the estimation of $\mathcal{I}_2$. For later use we change the dilation factor r, which is the same in both directions, to r_1 in the x direction and r_2 in the y direction ($r_1, r_2 \in \mathbb{N}$). Then $\mathcal{I}_2$ corresponds simply to the case $r_1 = r = r_2$, and

$$\begin{aligned}
\mathcal{I}_2 &= \int_{2^{-N+r_1}}^{1} \int_{2^{-M+r_2}}^{1} \left| \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \widehat{a}(k,l)\phi(k,\ell) w_{k,\ell}(x,y) \right|^p dy\,dx \\
&\leqslant \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} \int_{2^{-N+r_1}}^{1} \int_{2^{-M+r_2}}^{1} \left| \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \widehat{a}(k,\ell)\phi(k,\ell) w_{k,\ell}(x,y) \right|^p dy\,dx \\
&= \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} \sum_{\mu=1}^{N-r_1} \sum_{\upsilon=1}^{M-r_2} \int_{2^{-\mu}}^{2^{-\mu+1}} \int_{2^{-\upsilon}}^{2^{-\upsilon+1}} \left| \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \widehat{a}(k,\ell)\phi(k,\ell) w_{k,\ell}(x,y) \right|^p dy\,dx.
\end{aligned}$$

Using Hölder's inequality we obtain

$$\mathcal{I}_2 \leqslant \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} \sum_{\mu=1}^{N-r_1} \sum_{\upsilon=1}^{M-r_2} 2^{\mu(p-1)} 2^{\upsilon(p-1)} \left(\int_{2^{-\mu}}^{2^{-\mu+1}} \int_{2^{-\upsilon}}^{2^{-\upsilon+1}} \left| \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \widehat{a}(k,\ell)\phi(k,\ell) w_{k,\ell}(x,y) \right| dy\,dx \right)^p.$$

Set $K_\phi^{(i,j)} = \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \phi(k,\ell) w_{k,\ell}$, and consider the integral

$$\begin{aligned}
I_{\mu,\nu} &= \int_{2^{-\mu}}^{2^{-\mu+1}} \int_{2^{-\upsilon}}^{2^{-\upsilon+1}} \left| \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \widehat{a}(k,\ell)\phi(k,\ell) w_{k,\ell}(x,y) \right| dy\,dx \\
&= \int_0^1 \int_0^1 \chi_{[2^{-\mu},2^{-\mu+1})\times[2^{-\nu-1},2^{-\nu})}(x,y) \left| \left(a * K_\phi^{(i,j)}\right)(x,y) \right| dy\,dx.
\end{aligned}$$

Recall, $\operatorname{supp} a \subset [0, 2^{-N}) \times [0, 2^{-M})$, and notice that $\mu \leqslant N$, $\nu \leqslant M$, and $0 \leqslant s < 2^{-N}$, $0 \leqslant t < 2^{-M}$ imply

$$\chi_{[2^{-\mu},2^{-\mu+1})\times[2^{-\nu-1},2^{-\nu})}(x,y) = \chi_{[2^{-\mu},2^{-\mu+1})\times[2^{-\nu-1},2^{-\nu})}(x \dotplus s, y \dotplus t).$$

Thus

$$\begin{aligned}
&\chi_{[2^{-\mu},2^{-\mu+1})\times[2^{-\nu-1},2^{-\nu})}(x,y) \left| \left(a * K_\phi^{(i,j)}\right)(x,y) \right| \\
&\quad = \left| \int_0^{2^{-N}} \int_0^{2^{-M}} a(s,t) \chi_{[2^{-\mu},2^{-\mu+1})\times[2^{-\nu-1},2^{-\nu})}(x \dotplus s, y \dotplus t) K_\phi^{(i,j)}(x \dotplus s, y \dotplus t)\,dt\,ds \right| \\
&\quad = \left| \left(a * \left(\chi_{[2^{-\mu},2^{-\mu+1})\times[2^{-\nu-1},2^{-\nu})} K_\phi^{(i,j)}\right)\right)(x,y) \right|.
\end{aligned}$$

Hence

$$I_{\mu,\nu} = \left\| a * \left(\chi_{[2^{-\mu},2^{-\mu+1})\times[2^{-\nu-1},2^{-\nu})} K_\phi^{(i,j)}\right) \right\|_1 \leqslant \|a\|_1 \int_{2^{-\mu}}^{2^{-\mu+1}} \int_{2^{-\upsilon}}^{2^{-\upsilon+1}} \left| K_\phi^{(i,j)}(x,y) \right| dy\,dx.$$

By the Cauchy–Schwarz inequality, we have

$$\|a\|_1 \leqslant \int_0^{2^{-N}} \int_0^{2^{-M}} \left| a(s,t) \right| ds\,dt \leqslant 2^{-(N+M)/2} \|a\|_2 \leqslant 2^{-(N+M)/2} 2^{-(N+M)(1/2-1/p)} = 2^{(N+M)(1/p-1)}.$$

Consequently,

$$I_{\mu,\nu} \leqslant 2^{(N+M)(1/p-1)} \int\limits_{2^{-\mu}}^{2^{-\mu+1}} \int\limits_{2^{-\upsilon}}^{2^{-\upsilon+1}} \left|K_\phi^{(i,j)}(x,y)\right| dy\, dx.$$

Substituting this estimate into $\mathcal{I}_2$ we have

$$\mathcal{I}_2 \leqslant 2^{(N+M)(1-p)} \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} \sum_{\mu=1}^{N-r_1} \sum_{\upsilon=1}^{M-r_2} 2^{\mu(p-1)} 2^{\upsilon(p-1)} \left(\int\limits_{2^{-\mu}}^{2^{-\mu+1}} \int\limits_{2^{-\upsilon}}^{2^{-\upsilon+1}} \left|K_\phi^{(i,j)}(x,y)\right| dy\, dx \right)^p. \tag{3.2}$$

If $k = 2^i, 2^{i+1}$, or $\ell = 2^j, 2^{j+1}$ then the support of $D_{k,\ell}(x,y) = D_k(x)D_\ell(y)$ is disjoint from the range of integration. Therefore $K_\phi^{(i,j)}(x,y)$ takes on the following simple form after partial summation

$$K_\phi^{(i,j)}(x,y) = \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \phi(k,\ell) w_{k,\ell}(x,y) = \sum_{k=2^i+1}^{2^{i+1}-1} \sum_{\ell=2^j+1}^{2^{j+1}-1} D_{k,\ell}(x,y)\big(\phi(k-1,\ell-1) - \phi(k,\ell-1)$$
$$- \phi(k-1,\ell) + \phi(k,\ell)\big) \quad \big(2^{-\mu} \leqslant x < 2^{-\mu+1},\ 2^{-\upsilon} \leqslant x < 2^{-\upsilon+1}\big).$$

Applying first the Sidon inequality in (ii) of Lemma 3.1 to the integrals and then the M–H–M condition gives the estimate

$$\begin{aligned}
&\int\limits_{2^{-\mu}}^{2^{-\mu+1}} \int\limits_{2^{-\upsilon}}^{2^{-\upsilon+1}} \left|K_\phi^{(i,j)}(x,y)\right| dy\, dx \\
&\quad = \int\limits_{2^{-\mu}}^{2-\mu+1} \int\limits_{2^{-\upsilon}}^{2^{-\upsilon+1}} \left| \sum_{k=2^i+1}^{2^{i+1}-1} \sum_{\ell=2^j+1}^{2^{j+1}-1} D_{k,\ell}(x,y)\big(\phi(k-1,\ell-1) - \phi(k,\ell-1) - \phi(k-1,\ell) + \phi(k,\ell)\big) \right| dy\, dx \\
&\quad \leqslant C_p 2^{(\mu+\nu)(1-1/q)} \left(\sum_{k=2^i}^{2^{i+1}-2} \sum_{\ell=2^j}^{2^{j+1}-2} \left|\phi(k,\ell) - \phi(k+1,\ell) - \phi(k,\ell+1) + \phi(k+1,\ell+1)\right|^q \right)^{1/q} \\
&\quad \leqslant C_p 2^{(\mu+\nu)(1-1/q)} 2^{-(i+j)(1-1/q)}.
\end{aligned}$$

Then we can continue (3.2) as follows

$$\begin{aligned}
\mathcal{I}_2 &\leqslant C_p 2^{(N+M)(1-p)} \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} 2^{-(i+j)(1-1/q)p} \sum_{\mu=1}^{N-r_1} \sum_{\upsilon=1}^{M-r_2} 2^{(\mu+\nu)(2p-1-p/q)} \\
&\leqslant C_p 2^{(N+M)(1-p)} 2^{(N+M)(p/q-p)} 2^{(N+M-(r_1+r_2))(2p-1-p/q)} \\
&= C_p 2^{-(r_1+r_2)(2p-1-p/q)}
\end{aligned}$$

as $2p-1-p/q > 0$ or $q > \frac{p}{2p-1}$. Recall that $r_1 = r = r_2$ in this case, i.e.

$$\mathcal{I}_2 \leqslant C_p 2^{-2r(2p-1-p/q)}. \tag{3.3}$$

We note that this is the same inequality obtained as in the one-dimensional case. This is due to the fact set (2) contains no neighborhood of 0. This is the desired estimate to show T_ϕ is p-quasilocal with $\delta = 2(2p-1-p/q)$.

Actually we proved that

$$\int\limits_{[2^{-N+r_1},1)\times[2^{-M+r_2},1)} |T_\phi a|^p \leqslant C_p 2^{-(r_1+r_2)(2p-1-p/q)}. \tag{3.4}$$

For the integrals $\mathcal{I}_1$ and $\mathcal{I}_3$ the argument is more delicate than the last and requires showing the uniform boundedness of a family of operators on L^2. This requires Lemmas 1 and 2. We provide only the proof for $\mathcal{I}_3$ as the one for $\mathcal{I}_1$ follows from symmetry. The range of integration in $\mathcal{I}_3$ will be split as follows

$$\big[2^{-N+r},1\big) \times \big[0,2^{-M+r}\big) = \big[2^{-N+r},1\big) \times \big[0,2^{-M}\big) \cup \big[2^{-N+r},1\big) \times \big[2^{-M},2^{-M+r}\big).$$

Then

$$\int\limits_{[2^{-N+r},1)\times[2^{-M},2^{-M+r})} |T_\phi a|^p \leqslant C_p 2^{-r(2p-1-p/q)} \tag{3.5}$$

follows from (3.4) with the choice $r_1 = r$, and $r_2 = 0$.

Set

$$\mathcal{I}_{3,K,L} = \int\limits_{[2^{-N+r},1)\times[0,2^{-M})} \left|S_{2^K,2^L} T_\phi(a)\right|^p. \tag{3.6}$$

For $\mathcal{I}_{3,K,L}$,

$$\begin{aligned}\mathcal{I}_{3,K,L} &= \int\limits_{2^{-N+r}}^{1} \int\limits_{0}^{2^{-M}} \left| \left(a * \sum_{k=2^N}^{2^K-1} \sum_{\ell=2^M}^{2^L-1} \phi(k,\ell) w_{k,\ell} \right)(x,y) \right|^p dy\, dx \\ &\leqslant \sum_{i=N}^{K-1} \int\limits_{2^{-N+r}}^{1} \int\limits_{0}^{2^{-M}} \left| \int\limits_{0}^{2^{-N}} \int\limits_{0}^{2^{-M}} a(s,t) \sum_{l=2^M}^{2^L-1} \left(\sum_{k=2^i}^{2^{i+1}-1} \phi(k,\ell) w_k(x \dotplus s) \right) w_\ell(y \dotplus t)\, dt\, ds \right|^p dy\, dx.\end{aligned}$$

Let $\beta_{i,x\dotplus s}(\ell) = \sum_{k=2^i}^{2^{i+1}-1} \phi(k,\ell) w_k(x \dotplus s)$, and $a_s(t) = a(s,t)$. Then by these notations and by using Hölder's inequality for y with exponent $\frac{1}{p}$ we obtain

$$\begin{aligned}\mathcal{I}_{3,K,L} &\leqslant 2^{M(p-1)} \sum_{i=N}^{K-1} \int\limits_{2^{-N+r}}^{1} \left(\int\limits_{0}^{2^{-N}} \int\limits_{0}^{2^{-M}} \left| \int\limits_{0}^{2^{-M}} a_s(t) \sum_{\ell=2^M}^{2^L-1} \beta_{i,x\dotplus s}(\ell) w_l(y \dotplus t)\, dt \right| dy\, ds \right)^p dx \\ &= 2^{M(p-1)} \sum_{i=N}^{K-1} \int\limits_{2^{-N+r}}^{1} \left(\int\limits_{0}^{2^{-N}} \int\limits_{0}^{2^{-M}} \left| (T_{\beta_{i,x\dotplus s}} a_s)(y) \right| dy\, ds \right)^p dx.\end{aligned}$$

Let us split the outer integral and use the Cauchy–Schwarz-inequality for y followed by Hölder's inequality for s with exponent $\frac{1}{p}$. Then

$$\begin{aligned}\mathcal{I}_{3,K,L} &\leqslant 2^{M(p-1)} \sum_{i=N}^{K-1} \sum_{\mu=1}^{N-r} \int\limits_{2^{-\mu}}^{2^{-\mu+1}} \left(\int\limits_{0}^{2^{-N}} 2^{-M/2} \|T_{\beta_{i,x\dotplus s}} a_s\|_2\, ds \right)^p dx \\ &\leqslant 2^{M(p/2-1)} \sum_{i=N}^{K-1} \sum_{\mu=1}^{N-r} 2^{\mu(p-1)} \left(\int\limits_{0}^{2^{-N}} \int\limits_{2^{-\mu}}^{2^{-\mu+1}} \|T_{\beta_{i,x\dotplus s}} a_s\|_2\, dx\, ds \right)^p.\end{aligned} \tag{3.7}$$

Expanding $T_{\beta_{i,x\dotplus s}} a_s$ into a Walsh–Fourier series and computing the l^2-norm of this series, we obtain

$$\int\limits_{2^{-\mu}}^{2^{-\mu+1}} \|T_{\beta_{i,x\dotplus s}} a_s\|_2\, dx \leqslant 2^{-\mu/2} \left(\int\limits_{2^{-\mu}}^{2^{-\mu+1}} \|T_{\beta_{i,x\dotplus s}} a_s\|_2^2\, dx \right)^{1/2} = 2^{-\mu/2} \left(\sum_{\ell=2^M}^{2^L-1} \left|\widehat{a}_s(\ell)\right|^2 \int\limits_{2^{-\mu}}^{2^{-\mu+1}} \left|\beta_{i,x\dotplus s}(\ell)\right|^2 dx \right)^{1/2}. \tag{3.8}$$

Thus we need to bound

$$\int\limits_{2^{-\mu}}^{2^{-\mu+1}} \left|\beta_{i,x\dotplus s}(\ell)\right|^2 dx = \int\limits_{2^{-\mu}}^{2^{-\mu+1}} \left| \sum_{k=2^i}^{2^{i+1}-1} \phi(k,\ell) w_k(x \dotplus s) \right|^2 dx = \int\limits_{2^{-\mu}}^{2^{-\mu+1}} \left| \sum_{k=2^i}^{2^{i+1}-1} \phi(k,\ell) w_k(x) \right|^2 dx,$$

where the last equality arises from the use of translation invariance since $\mu \leqslant N$ and $0 \leqslant s < 2^{-N}$. By summation by parts we have

$$\begin{aligned}\sum_{k=2^i}^{2^{i+1}-1} \phi(k,\ell) w_k(x) &= w_{2^i}(x) \sum_{k=0}^{2^i-1} \phi\big(k+2^i,\ell\big) w_k(x) = w_{2^i} \sum_{k=1}^{2^i-1} \big(\phi\big(2^i+k-1,\ell\big) - \phi\big(2^i+k,\ell\big)\big) D_k(x) \\ &\quad + \phi\big(2^{i+1}-1,\ell\big) D_{2^i}(x).\end{aligned}$$

The relation $i \geqslant N > \mu$ implies that D_{2^i} is 0 on $[2^{-\mu}, 2^{-\mu+1})$. Therefore the use of the Sidon type inequality (iii) of Lemma 3.1 and an M–H–M condition give

$$\int_{2^{-\mu}}^{2^{-\mu+1}} \left|\beta_{i,x\dot{+}s}(\ell)\right|^2 dx = \int_{2^{-\mu}}^{2^{-\mu+1}} \left|\sum_{k=1}^{2^i-1} \left(\phi\left(2^i+k-1,\ell\right)-\phi\left(2^i+k,\ell\right)\right)D_k(x)\right|^2 dx$$
$$\leqslant C2^{\mu(3-2/q)}\left(\sum_{k=2^i}^{2^{i+1}-1}\left|\phi(k,\ell)-\phi(k+1,\ell)\right|^q\right)^{2/q} \leqslant C2^{\mu(3-2/q)}2^{-2i(1-1/q)}.$$

Replacing this in (3.8) yields

$$\int_{2^{-\mu}}^{2^{-\mu+1}} \|T_{\beta_{i,x\dot{+}s}} a_s\|_2\, dx \leqslant 2^{-\mu/2}C2^{\mu(3/2-1/q)}2^{-i(1-1/q)}\left(\sum_{\ell=2^M}^{2^L-1}\left|\widehat{a}_s(\ell)\right|^2\right)^{1/2} \leqslant C2^{\mu(1-1/q)}2^{-i(1-1/q)}\|a_s\|_2.$$

Then we can continue (3.7) as follows

$$\mathcal{I}_{3,K,L} \leqslant C2^{M(p/2-1)}\sum_{i=N}^{K-1}2^{-ip(1-1/q)}\sum_{\mu=1}^{N-r}2^{\mu(2p-1-p/q)}\left(\int_0^{2^{-N}}\|a_s\|_2\, ds\right)^p.$$

Since a is an atom we have

$$\int_0^{2^{-N}}\|a_s\|_2\, ds \leqslant 2^{-N/2}\left(\int_0^{2^{-N}}\int_0^{2^{-M}}\left|a(s,t)\right|^2 dt\, ds\right)^{1/2} \leqslant 2^{-N/2}2^{(M+N)(1/p-1/2)}.$$

Consequently,

$$\mathcal{I}_{3,K,L} \leqslant C2^{M(p/2-1)}\sum_{i=N}^{K-1}2^{-ip(1-1/q)}\sum_{\mu=1}^{N-r}2^{\mu(2p-1-p/q)}2^{-Np/2}2^{(M+N)(1-p/2)}$$
$$= C2^{N(1-p)}\sum_{i=N}^{K-1}2^{-ip(1-1/q)}\sum_{\mu=1}^{N-r}2^{\mu(2p-1-p/q)} \leqslant C2^{-r(2p-1-p/q)}$$

provided $(2p-1-p/q) > 0$. This requires $p > \frac{q}{2q-1}$. We note that the definition of $\mathcal{I}_{3,K,L}$ (see (3.6)) and (3.1) imply

$$\int_{[2^{-N+r},1)\times[0,2^{-M})} \left|T_\phi(a)\right|^p \leqslant C2^{-r(2p-1-p/q)}. \tag{3.9}$$

This is the desired estimate with $\delta = (2p-1-p/q) > 0$.

Combining (3.3), (3.5), and (3.9) for $\mathcal{I}_i$ $(i=1,2,3)$ we have shown T_ϕ is p-quasilocal with $\delta = 2p-1-p/q$ for $p > \frac{q}{2q-1}$ when $1 < q \leqslant 2$. □

In the case the multiplier factors into two one-dimensional multipliers such that each satisfies the one-dimensional M–H–M condition, then the proof simplifies significantly. As an example, the two-dimensional Sunouchi multiplier factors in such a manner. We include Theorem 3.3 for completeness.

Theorem 3.3. *If the bounded multiplier ϕ factors as two one-dimensional multipliers that satisfy the one-dimensional Marcinkiewicz–Hörmander–Mihlin condition* (2.1) *for some q, $1 < q \leqslant 2$, and p satisfies $\frac{q}{2q-1} < p \leqslant 1$, then the multiplier operator T_ϕ is bounded on H^p.*

Proof. Suppose the multiplier ϕ factors as $\phi(k,l) = \lambda(k)\beta(l)$ and λ and β satisfy the one-dimensional M–H–M condition. The argument for each of the three sets (1)–(3) from the proof of Theorem 2.1 that compose the complement of the identity are similar. We include the main part for the strip (3) that contains a neighborhood of zero. We have

$$\mathcal{I}_3 = \int_{2^{-N+r}}^{1}\int_0^{2^{-M}}\left|T_\phi(a)\right|^p = \int_{2^{-N+r}}^{1}\int_0^{2^{-M}}\left|\sum_{i=N}^{\infty}\sum_{j=M}^{\infty}\sum_{k=2^i}^{2^{i+1}-1}\sum_{\ell=2^j}^{2^{j+1}-1}\int_0^{2^{-N}}\int_0^{2^{-M}} a(s,t)\phi(k,\ell)w_{k,\ell}(x\dot{+}s, y\dot{+}t)\, dt\, ds\right|^p dy\, dx. \tag{3.10}$$

Continuing,

$$\mathcal{I}_3 = \int_{2^{-N+r}}^{1} \int_{0}^{2^{-M}} \left| \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} \sum_{\ell=2^j}^{2^{j+1}-1} \int_0^{2^{-N}} \int_0^{2^{-M}} a(s,t)\lambda(k)\beta(\ell) w_k(x \dotplus s) w_\ell(y \dotplus t)\,dt\,ds \right|^p dy\,dx$$

$$\leqslant C2^{M(p-1)} \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \left(\int_0^{2^{-N}} \left| \sum_{k=2^i}^{2^{i+1}-1} \lambda(k) w_k(x \dotplus s) \right| \int_0^{2^{-M}} \left| \int_0^{2^{-M}} a(s,t) \sum_{j=M}^{\infty} \sum_{\ell=2^j}^{2^{j+1}-1} \beta(\ell) w_\ell(y \dotplus t)\,dt \right| dy\,ds \right)^p dx. \tag{3.11}$$

Now taking the inner t, y integrals and using Hölder's inequality

$$\int_0^{2^{-M}} \left| \int_0^{2^{-M}} a(s,t) \sum_{j=M}^{\infty} \sum_{\ell=2^j}^{2^{j+1}-1} \beta(\ell) w_\ell(y \dotplus t)\,dt \right| dy \leqslant 2^{-M/2} \left(\int_0^1 \left| \int_0^{2^{-M}} a(s,t) \sum_{j=M}^{\infty} \sum_{\ell=2^j}^{2^{j+1}-1} \beta(\ell) w_\ell(y \dotplus t)\,dt \right|^2 dy \right)^{1/2}$$

$$= 2^{-M/2} \|T_\beta a_s\|_2 \leqslant 2^{-M/2} \sup\left|\beta(l)\right| \|a_s\|_2 \leqslant 2^{-M/2} \|a_s\|_2.$$

Substituting this estimate back into (3.11) and using translation invariance in the x-integral, we have

$$\int_{2^{-N+r}}^{1} \int_0^{2^{-M}} \left|T_\phi(a)\right|^p \leqslant C2^{M(p-1)} 2^{-Mp/2} \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \left(\int_0^{2^{-N}} \left| \sum_{k=2^i}^{2^{i+1}-1} \lambda(k) w_k(x+s) \|a_s\|_2 \right| ds \right)^p dx$$

$$\leqslant C2^{M(p-1)} 2^{-Mp/2} \left(\int_0^{2^{-N}} \|a_s\|_2\,ds \right)^p \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \left| \sum_{k=2^i}^{2^{i+1}-1} \lambda(k) w_k(x) \right|^p dx.$$

From Hölder's inequality and the fact a is a rectangular atom,

$$\left(\int_0^{2^{-N}} \|a_s\|_2\,ds \right)^p \leqslant 2^{-Np/2} \left(\int_0^{2^{-N}} \|a_s\|_2^2\,ds \right)^{p/2} \leqslant 2^{-Np/2} \|a\|_2^p \leqslant 2^{-Np/2} 2^{p(N+M)(1/p-1/2)}.$$

Using this estimate, we have

$$\int_{2^{-N+r}}^{1} \int_0^{2^{-M}} \left|T_\phi(a)\right|^p \leqslant C2^{N(1-p)} \sum_{i=N}^{\infty} \sum_{n=0}^{N-r-1} 2^{n(p-1)} \left(\int_{2^{-n-1}}^{2^{-n}} \left| \sum_{k=2^i}^{2^{i+1}-1} \lambda(k) w_k(x) \right| dx \right)^p$$

$$\leqslant C2^{N(1-p)} \sum_{i=N}^{\infty} \sum_{n=0}^{N-r-1} 2^{n(p-1)} 2^{np(1-1/q)} 2^{ip(1/q-1)}$$

$$\leqslant C2^{N(1-p)} \sum_{i=N}^{\infty} 2^{ip(1/q-1)} \sum_{n=0}^{N-r-1} 2^{n(2p-1-p/q)}$$

$$\leqslant C2^{-r(2p-1-p/q)},$$

which is the desired estimate as $2p-1-p/q > 0$. □

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